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SOME UNIVERSAL CONSTRUCTIONS IN ABSTRACT TOPOLOGICAL DYNAMICS

Vladimir Pestov

Victoria University of Wellington, Wellington, New Zealand

Dedicated to Professor Robert Ellis on the occasion of his retirement

Abstract. This small survey of basic universal constructions related to the actions of topological groups on compacta is centred around a new result — an intrinsic description of extremely amenable topological groups (i.e., those having a fixed point in each compactum they act upon), solving a 1967 problem by Granirer. Another old problem whose solution (in the negative) is noted here, is a 1957 conjecture by Teleman on irreducible representations of general topological groups. Our exposition covers the greatest ambit and the universal minimal flow, as well as closely related constructions and results from representation theory. We present in a simplified fashion the known examples of extremely amenable groups and discuss their relationship with a 1969 problem by Ellis. Open questions, including those from bordering disciplines, are reviewed along the way.

0. Introduction

The principal object of study in abstract topological dynamics is a G-flow (topological transformation group, topological dynamical system), that is, a triple $\mathfrak{X} = (G, X, \tau)$ consisting of a Hausdorff topological group G, a Tychonoff topological space X, and a continuous action $\tau: G \times X \to X$. Theory is at its most advanced for compact phase spaces X, in which case τ can be thought of as a continuous homomorphism $G \to \operatorname{Homeo} X$, where the group of self-homeomorphisms is equipped with the compact-open topology. It is therefore most natural to begin a journey into the realm of abstract topological dynamics by asking: does *every* topological group G admit an effective action on a compact space, thereby allowing for a dynamically substantial theory?

This fundamental question was answered in the positive forty years ago by Teleman [T] who produced a fine collection of results on the crossroads of topological dynamics with representation theory and theory of C^*-algebras. We present those results, leading to the concept of the greatest ambit, in Sections 1 and 4-5. Among the later corollaries, is a 1986 Uspenskiĭ's observation [U] of the existence of a universal Polish group, answering the 'right' version of Schreier–Ulam's question from the Scottish Book (Section 3). We also point out, possibly for the first time, that a

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1957 conjecture of Teleman can be refuted through combining some developments in operator theory and in abstract harmonic analysis (Section 2).

Of all compact G-flows, minimal G-flows (that is, as name suggests, those having no proper compact G-subflows) are of particular importance. In Section 6 we introduce the universal minimal flow, $\mathcal{M}(G)$, for any topological group G. The next important question to address is that of the 'size' of the universal minimal flow. In particular, when is $\mathcal{M}(G)$ nontrivial, and when is the action of G upon $\mathcal{M}(G)$ effective?

A topological group G such that the universal (therefore, every) minimal G-flow is a singleton is called *extremely amenable* [Gr1, Mi, Gr2] (or else a group having the *fixed point on compacta property* [G3]).

The very existence of such groups was at first an open problem, though topological semigroups with a similar property were known to exist in abundance [Gr1, Mi]. By now there are known examples of extremely amenable topological groups of at least three different kinds [HR, G3, P]. In 1967 Granirer [Gr1] suggested the problem of describing extremely amenable topological groups in intrinsic terms. We propose such a description in Sections 7 and 8. Recall that a subset S of a topological group G is *left syndetic* if $KS = G$ for some compact $K \subseteq G$. We show that a topological group G is extremely amenable if and only if for every left syndetic subset $S \subseteq G$ the set SS^{-1} is everywhere dense in G. (One can also replace 'left syndetic' with 'big on the left,' that is, left syndetic in the discrete topology.)

Employing this new criterion, we sketch the known examples of extremely amenable groups. Historically the first such example, constructed by Herer and Christensen, utilises the existence of abelian topological groups without nontrivial unitary representations (see Section 13). Another source of examples, presented in Section 9, was discovered by Glasner and (independently) Furstenberg and Weiss, who built up on the work of Gromov and Milman, and it includes the (even monothetic) group of measurable maps from I to the unit circle, equipped with the L_1-metric. Finally, among the extremely amenable groups proposed by the present author (Section 10), is the group $\mathrm{Homeo}_+(I)$ of orientation-preserving homeomorphisms of the unit interval. We review a number of dynamical consequences for groups having importance in Analysis and observe that beyond the locally compact case, amenability is not inherited by closed subgroups (Section 10).

The existence of extremely amenable groups answers in the negative a 1969 question by Ellis from [E3], which was largely responsible for research in the area and can be reformulated in equivalent terms as follows: are points of the greatest ambit $\mathcal{S}(G)$ of a topological group G separated by continuous equivariant maps to minimal G-flows? This problem and related developments form the subject of Section 11.

Our criterion of extreme amenability is accompanied by a satellite result describing, also for the first time, those topological groups G admitting effective minimal actions on compacta (Section 12). A topological group G acts effectively on $\mathcal{M}(G)$ if and only if for every $g \in G$, $g \neq e$ there is a left syndetic set S with $x \notin \overline{SS^{-1}}$. We derive from this criterion an easy proof of the fact that every locally compact group acts effectively on the universal minimal flow. This is of course a corollary of a stronger theorem by Veech [V2] stating that every locally compact group acts *freely* on a compactum — the result of which a 'soft' proof has so far escaped us.

Finally, in Section 13 we discuss a few results and questions related to the old and still open problem of describing those topological groups admitting a complete

system of strongly continuous unitary representations in Hilbert spaces.

Our usage of basic concepts from topology, functional analysis, topological algebra, representation theory, and topological dynamics is fairly standard, while our list of bibliography is kept to a bare minimum.

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1. Flows and Representations

Topological dynamics is intimately linked to representation theory, as can be seen from the following observation. Let ρ be a representation of an abstract group G in a Banach space E by isometries, that is, $\rho\colon G\to \mathrm{Iso}\,(E)$ is an homomorphism to the group of all linear isometries of E onto itself. For every $g\in G$ the dual operator, $\rho_g^*\colon E^*\to E^*$, to the isometry $\rho_g\colon E\to E$ is an isometry again, and the restriction, ρ_g^B, of $\rho^*_{g^{-1}}$ to the unit ball, B^*_w, of E^*, equipped with the w^*-topology (the coarsest topology making each evaluation map $\hat{x}\colon E^*\to\mathbb{K}$, $\hat{x}(f)=f(x)$, continuous), is a self-homeomorphism. It is easy to check that the resulting map

$$\rho^B\colon G\ni g\mapsto \rho^B_g\in \mathrm{Homeo}\,(B^*_w)$$

is a group homomorphism, that is, an action of G on the compact space B^*_w.

Now assume that G is a topological group. Recall that a representation $\rho\colon G\to \mathrm{GL}(E)$ (in particular, a representation $\rho\colon G\to\mathrm{Iso}\,(E)$) is *strongly continuous* if it is continuous with respect to the topology on $\mathrm{GL}(E)$ inherited from the Tychonoff product topology on E^E. (This, the so-called *strong operator topology,* though not a group topology on $\mathrm{GL}(E)$, becomes a group topology when restricted to $\mathrm{Iso}\,(E)$.)

PROPOSITION 1.1 (Teleman [T]). (i) *If ρ is strongly continuous then ρ^B is continuous.*

(ii) *If ρ is faithful then ρ^B is effective.*

(iii) *If $\rho\colon G\hookrightarrow \mathrm{Iso}\,(E)$ is an embedding of topological groups, then so is $\rho^B\colon G\hookrightarrow \mathrm{Homeo}\,(B^*_w)$.*

PROOF. Let us compare standard basic neighbourhoods of identity in the topological group $\mathrm{Iso}\,(E)$,

$$\{u\in\mathrm{Iso}\,(E)\colon \forall i=1,2,\dots,n,\ \|\,u(x_i)-x_i\,\|<\varepsilon\},\ x_i\in E,\ \varepsilon>0,$$

and in $\mathrm{Homeo}\,(B^*_w)$,

$$\{u\in\mathrm{Homeo}\,(B^*_w)\colon \forall f\in B^*_w,\ (u(f),f)\in V\},\ V\in\mathcal{U}$$

where $\mathcal{U}_X$ is the (unique) compatible uniformity on B_w^*. Applying the Hahn–Banach theorem in the first case and noticing that $\mathcal{U}_X$ is induced by the additive uniformity on E^* equipped with the weak* topology in the second case, we reduce both neighbourhoods to the same form:

$$\{u\colon \forall i = 1,2,\dots,n,\ \forall f \in B_w^*,\ |f(u(x_i)) - f(x_i)| < \varepsilon\},\ x_i \in E,\ \varepsilon > 0$$

It implies that $\mathrm{Iso}\,(E)$ canonically embeds into $\mathrm{Homeo}\,(B_w^*)$ as a topological subgroup, whence all three statements follow. $\square$

To conclude that every topological group G acts effectively on a compact space, it is enough to observe the following.

THEOREM 1.2 (Teleman [T]). *Every topological group G admits a faithful strongly continuous representation ρ by isometries in a suitable Banach space E. Moreover, $\rho\colon G \to \mathrm{Iso}\,(E)$ is in this case a topological embedding.*

PROOF. We avoid using the terms 'left' and 'right' uniform structure altogether because there is a considerable confusion surrounding their usage. Instead we denote, once and for all, by $\mathcal{U}_{\Rsh}(G)$ the uniform structure on G whose base is formed by the entourages of the diagonal

$$V_{\Rsh} = \{(x,y)\colon xy^{-1} \in V\},$$

where V is a neighbourhood of e_G. We say that a function $f\colon G \to \mathbb{C}$ is $\mathcal{U}_{\Rsh}$*-uniformly continuous,* or $\mathcal{U}_{\Rsh}$*-u.c.,* if

$$\forall \varepsilon > 0,\ \exists V \in \mathcal{N}(G),\ \forall x,y \in G,\ xy^{-1} \in V \Rightarrow |f(x) - f(y)| < \varepsilon \qquad (*)$$

The uniformity $\mathcal{U}_{\Lsh}$ and $\mathcal{U}_{\Lsh}$-uniformly continuous functions on a topological group are introduced in a similar way, though they are never used in this article.

The totality of all bounded complex-valued $\mathcal{U}_{\Rsh}$-u.c. functions on a topological group G equipped with the supremum norm and pointwise operations is easily checked to form a commutative C^*-algebra — indeed a C^*-subalgebra of $C(\beta G) \cong l_\infty(|G|)$ — which we denote by $C^*_{\Rsh}(G)$.

Every element $g \in G$ determines a left shift $L_g\colon C^*_{\Rsh}(G) \to C^*_{\Rsh}(G)$ via

$$L_g(f)(h) = f(g^{-1}h)$$

The emerging representation $L\colon G \to \mathrm{Iso}\,C^*_{\Rsh}(G)$ is strongly continuous: if V is symmetric and as in $(*)$, then $\|L_g f - f\| < \varepsilon$ whenever $g \in V$. Moreover, L is a topological embedding: given a symmetric neighbourhood of identity, V, one can construct a $\mathcal{U}_{\Rsh}$-u.c. bounded $f\colon G \to \mathbb{C}$ with $f \equiv 0$ outside V and $f(e) = 1$; the set $W = \{u \in \mathrm{Iso}\,C^*_{\Rsh}(G)\colon \|u(f) - f\| < 1\}$ is open in the strong topology and $e \in W \cap G \subseteq V$. $\square$

It seems that Teleman's results later became a part of the folklore and were even sometimes reproduced with proofs without referring to the original

2. Teleman's Conjecture on Irreducible Banach Representations

Having in view the celebrated 1943 Gelfand–Raĭkov theorem on the existence of a complete system of irreducible unitary representations for any locally compact group, Teleman asked in his 1957 paper [T] whether every topological group admits a complete system of strongly continuous irreducible representations in Banach spaces by isometries. A negative answer to this question can be deduced from more recent results obtained in operator theory and abstract harmonic analysis, but it is not apparent if it was ever noticed by anyone.

Recall that a topological group is called *minimally almost periodic* if it admits no nontrivial finite-dimensional unitary representations, and *monothetic* if it contains an everywhere dense cyclic subgroup.

PROPOSITION 2.1. *A monothetic minimally almost periodic topological group G admits no nontrivial irreducible strongly continuous representations in a complex Banach space by isometries.*

PROOF. Assume the existence of such a representation, ρ. Let x be a generator of an everywhere dense cyclic subgroup of G. The operator ρ_x is then an isometry of the complex Banach space of representation, E, and therefore, as a well-known consequence of Riesz' theorem [B, th. 3.1], admits a nontrivial closed invariant subspace, F, whenever $\dim E > 1$. Since F remains invariant for all operators ρ_g, $g \in G$, one concludes that $\dim E = 1$ and the representation ρ, being a continuous character $G \to U(1)$, is trivial. $\square$

Since there are numerous known examples of minimally almost periodic monothetic topological groups (see e.g. [DPS]), Teleman's conjecture does not survive. However, it can be brought back to life through simply dropping the requirement 'by isometries.'

QUESTIONS 2.2. (i) Does every topological group admit a complete system of strongly continuous irreducible representations in complex Banach spaces?

(ii) Assuming a topological group G admits a faithful strongly continuous representation in a Hilbert space, does it admit a complete system of irreducible representations in Hilbert spaces? $\boxed{?}$

If answered positively, the second question, applied to the topological group from Glasner–Furstenberg–Weiss's example (see Section 9 below), would provide a solution to the Invariant Subspace Problem.

3. Universal Group of Countable Weight

It was asked by Schreier and Ulam in the Scottish Book (problem 103 in the 1981 edition [Ma]) whether or not there existed a universal separable topological group, that is, a separable topological group G such that every other such group is isomorphic to a topological subgroup of G. The answer is 'no' because of simple set-theoretic reasons: while a separable group can have at most $\mathfrak{c}$ pairwise distinct subgroups, there are $\exp \mathfrak{c}$ pairwise non-isomorphic separable topological groups, cf. the comments on pp. 184–186 in [Ma]. However, the following elegant result seems to answer in the positive the 'right' version of Schreier–Ulam's question as suggested by Kallman (*ibid.*)

THEOREM 3.1 (Uspenskiĭ [U]). *The group* $\mathrm{Homeo}\,(I^{\aleph_0})$ *is a universal separable metrizable topological group. In other words, every separable metrizable topological group is isomorphic with a topological subgroup of* $\mathrm{Homeo}\,(I^{\aleph_0})$.

PROOF. If G is separable metrizable, then its points are separated from closed subsets by elements of a countable subset — and therefore by those of a separable Banach subspace, say E — of $C^*_r(G)$, and such a subspace can be also assumed invariant under left translations. The action of G on E determines an isometric embedding $G \hookrightarrow \mathrm{Iso}\,(E)$, and Proposition 1.1 yields an embedding of topological groups $G \hookrightarrow \mathrm{Homeo}\,(B^*_w)$. How it remains to apply Keller's theorem [BP]: all convex compact infinite-dimensional metrizable subsets of locally convex spaces are homeomorphic to the Hilbert cube $Q = I^{\aleph_0}$. □

4. Dynamics and C^*-Algebras

Let us cast a closer look at Teleman's construction. So far we made no use of the fact that $C^*_r(G)$ is a commutative unital C^*-algebra rather than just a Banach space, and the shifts L_g are C^*-algebra automorphisms and not merely isometries.

Assume for a while that a topological group G acts strongly continuously on a commutative unital C^*-algebra A by automorphisms. The Gelfand space, $\mathrm{Spec}_{\mathbb{C}}A$, formed by all multiplicative linear functionals $f\colon A \to \mathbb{C}$, is a closed subspace of the unit ball B^*_w, and it is clearly invariant under the action ρ^B of G. Therefore, G acts on $\mathrm{Spec}_{\mathbb{C}}A$ continuously. Going in the opposite direction, if X is a compact G-flow, then G acts strongly continuously by left shifts on the commutative C^*-algebra $C(X)$ of all complex-valued continuous functions on X, and $X \cong \mathrm{Spec}_{\mathbb{C}}C(X)$ as G-spaces. The functorial correspondence $A \mapsto \mathrm{Spec}_{\mathbb{C}}A$ is an anti-equivalence of categories.

THEOREM 4.1. *Let G be a topological group. The category of all compact G-flows and G-equivariant continuous maps is anti-equivalent to the category of all strongly continuous representations of G in commutative unital C^*-algebras by automorphisms and intertwining unital C^*-algebra morphisms.* □

This simple observation helps to explain why theory of strongly continuous actions of topological groups on *arbitrary* unital C^*-algebras by automorphisms is universally accepted as the adequate version of *non-commutative dynamics*, cf. [Gu].

5. The Greatest Ambit

For a topological group G, denote the Gelfand space $\mathrm{Spec}_{\mathbb{C}}C_r(G)$ by $\mathcal{S}(G)$. Each element $g \in G$ determines an evaluation functional, $\hat{g} \in \mathcal{S}(G)$, via $\hat{g}(f) = f(g)$. Since $\mathcal{U}_r$-u.c. functions separate points and closed subsets in a topological group, the resulting canonical map $G \ni g \mapsto \hat{g} \in \mathcal{S}(G)$ is a homeomorphic embedding. It is also equivariant with respect to the action of G on itself by left translations: $L^*_{g^{-1}}(\hat{x}) = \widehat{gx}$ for all $g, x \in G$. For this reason, we will be identifying each element $g \in G$ with $\hat{g} \in \mathcal{S}(G)$. Notice that the G-flow $\mathcal{S}(G)$ has a distinguished point, e. The G-orbit of e is everywhere dense in $\mathcal{S}(G)$, since the latter space is a compactification of G; it is, in fact, the Samuel compactification (see e.g. [En]) of the uniform space $(G, \mathcal{U}_r(G))$.

DEFINITION 5.1. A compact G-flow with a distinguished point, (X, x), is called a *G-ambit* if the orbit of x is everywhere dense in X. A *morphism* between G-ambits (X, x) and (Y, y) is a continuous G-equivariant map $X \to Y$ taking x to y. ▲

Let $X = (X, x)$ be an arbitrary G-ambit for a topological group G. Every $f \in C(X)$ determines a function $\tilde{f}\colon G \to \mathbb{C}$ via 'pullback:' $\tilde{f}(g) = f(gx)$. It is easy to see that $\tilde{f}$ is bounded and $\mathcal{U}_{\mathrm{r}}$-u.c. The C^*-algebra monomorphism $C(X) \hookrightarrow C^*_{\mathrm{r}}(G)$, determined by the correspondence $f \mapsto \tilde{f}$, intertwines the representations of G in $C(X)$ and $C^*_{\mathrm{r}}(G)$ and therefore determines, according to our earlier observation, a dual morphism of G-flows $\mathcal{S}(G) \to X$, which takes e to x and is therefore onto. (The image of $\mathcal{S}(G)$ is compact and therefore closed, and is at the same time everywhere dense since it contains the orbit $G \cdot x$.) We come to the following.

THEOREM 5.2 (Teleman)**.** *Every topological group G possesses the greatest ambit $\mathcal{S}(G) = (\mathcal{S}(G), e)$ in the sense that every G-ambit is the image of $\mathcal{S}(G)$ under a unique morphism of G-ambits. The greatest ambit contains G itself as an everywhere dense G-subspace.* □

As we will see later, the concept is very useful. It would be interesting to know the following.

QUESTION 5.3. Is there an analogue of the greatest ambit in noncommutative dynamics? $\boxed{?}$

EXAMPLE 5.4. Let us describe two 'diametrically opposite' situations where the structure of the greatest ambit is relatively well understood.

(i) If G is precompact, then $\mathcal{S}(G)$ is simply the compact topological group, $\widehat{G}$, completion of G, equipped with the left action of the subgroup G.

(ii) If G is discrete, then every complex-valued function on G is $\mathcal{U}_{\mathrm{r}}$-u.c. and as a corollary, $\mathcal{S}(G)$ is βG, the Stone-Čech compactification of G, equipped with the natural action of G on elements of βG (which can be identified with all ultrafilters on the set G) by left translations: if $g \in G$ and $U \in \beta G$, then $gU = \{gA\colon A \in U\} \in \beta G$. (See [A, E2, Vr].) ▲

The situation 'in between' is much less thoroughly understood. One significant recent advance is the description by Turek [Tu] of the greatest ambit, $\mathcal{S}(\mathbb{R})$, of the additive group of reals, $\mathbb{R}$, with the usual topology.

It only takes a slight modification of the arguments used in 1.1 and 1.2 to establish the following.

THEOREM 5.5. *The canonical action of a topological group G on its greatest ambit, $\mathcal{S}(G)$, determines an embedding of topological groups, $G \hookrightarrow$* Homeo$(\mathcal{S}(G))$. □

The terminology we suggest using on such occasions is that the action is *topologically effective.*

Presently the standard reference to the concept of the greatest ambit is Brook's paper [Br].

6. The Universal Minimal Flow

Let G be a topological group. It is clear that a compact G-flow, X, is minimal, that is, admits no proper compact G-subflows, if and only if the orbit of every $x \in X$ is everywhere dense in X. Since for an arbitrary compact G-flow the family of all compact non-empty G-subflows survives the intersections of chains, Zorn's lemma applies and yields the following result.

PROPOSITION 6.1. *Every compact G-flow contains a minimal G-subflow.* $\square$

A minimal G-subflow of the greatest ambit, $\mathcal{S}(G)$, is denoted by $\mathcal{M}(G)$ and called the *universal minimal G-flow.* The name is justified by the following result.

PROPOSITION 6.2. *For a topological group G every minimal G-flow, X, is the image of the universal minimal flow, $\mathcal{M}(G)$, under a surjective morphism of G-flows.*

PROOF. Let $x \in X$ be any. The restriction to (an isomorphic copy of) $\mathcal{M}(G)$ of the unique morphism of G-ambits $\mathcal{S}(G) \to X$ sending e to x has a compact (therefore closed) image in X, which image contains the everywhere dense G-orbit of x and thus coincides with X. $\square$

The ambiguity in choosing an $x \in X$ in the above proof is responsible for the fact that there is, generally speaking, no *canonical* morphism from $\mathcal{M}(G)$ onto X, unlike it is the case with the greatest ambit. This is also the reason why the proof of uniqueness is slightly more involved, and we refer the reader to e.g. [A].

PROPOSITION 6.3. *For a topological group G, the universal minimal G-flow is unique up to an isomorphism.* $\square$

What is the 'size' of the universal minimal flow for a given topological group? In particular, when is $\mathcal{M}(G)$ nontrivial and when G acts upon $\mathcal{M}(G)$ effectively? By far the strongest result in this direction to date is the following.

THEOREM 6.4 (Veech [V1])**.** *Every locally compact group G acts freely on the universal minimal flow.* $\square$

The Veech's proof is rather technical. While we are unable to find a 'soft' version of it, we offer a simple proof of the weaker statement: every locally compact group acts *effectively* on the universal minimal flow. This will be achieved by means of a new description of topological groups admitting effective minimal flows.

7. Syndetic Sets and The Size of Universal Minimal Flow

A subset S of a topological group G is called *left syndetic* if for some compact $K \subseteq G$ one has $KS = G$. If a set S is left syndetic with respect to the discrete topology on a group (that is, $FS = G$ for some F finite), then S is called *discretely left syndetic* or *big on the left.* If either G is abelian or S is symmetric then 'left' in both definitions can be dropped.

Here is how syndetic sets emerge in topological dynamics. Suppose a topological group G acts on a minimal compact G-flow X and let $x \in V \subseteq X$, where V is open. Then the open set $\widetilde{V} = \{g \in G\colon gx \in V\}$ is big on the left in G. Indeed, since the translations hV, $h \in G$ form an open cover of X (assuming a $y \in X$ is not in their union, the orbit of y would miss V, contrary to everywhere density in X), there is a finite subcover $\{fV\colon f \in F\}$, where $F \subseteq G$, $|F| < \infty$. It remains to notice that $\widetilde{fV} = f\widetilde{V}$ for each $f \in G$ and therefore $F\widetilde{V} = G$. (To see that $\widetilde{V}$ is not necessarily *right* syndetic, it is enough to study the full homeomorphism group acting on the circle $\mathbb{S}^1$.)

THEOREM 7.1. *Let G be a topological group and $g \in G$. The following are equivalent.*

(i) *The motion of the universal minimal flow $\mathcal{M}(G)$ determined by g is nontrivial, that is, $gx \neq x$ for some $x \in \mathcal{M}(G)$.*
(ii) *There is a left syndetic set $S \subseteq G$ with $g \notin \overline{SS^{-1}}$.*
(iii) *There is a set $S \subseteq G$ big on the left and such that $g \notin \overline{SS^{-1}}$.*
(iv) *There is an open set $S \subseteq G$ big on the left and such that $g \notin \overline{SS^{-1}}$.*

PROOF. (i) $\Rightarrow$ (iv): Let $\mathcal{U}_X$ denote the (unique) uniform structure on $X = \mathcal{M}(G)$. For every $E \in \mathcal{U}_X$, let $\tilde{E} = \{(g,h) \in G \times G \colon (gx, hx) \in E\}$. The collection $\{\tilde{E} \colon E \in \mathcal{U}_X\}$ forms a basis of a uniform structure, $\mathcal{V}$, on G contained in $\mathcal{U}_{r}(G)$.

Let $x \in X$ be such that $gx \neq x$. For some $E \in \mathcal{U}_X$ one has $E(x) \cap E(gx) = \emptyset$. Since $\mathcal{V} \subseteq \mathcal{U}_{r}(G)$, for a suitable symmetric $\mathcal{O} \in \mathcal{N}(G)$ and an $E_1 \in \mathcal{V}$, one has $\mathcal{O}_{r} \circ E_1 \subseteq E$ and therefore $\mathcal{O}E_1(x) \cap \mathcal{O}E_1(gx) = \emptyset$. Set $V = E_1(x) \cap g^{-1}E_1(gx)$; it is a neighbourhood of x and $\mathcal{O}V \cap gV \subseteq \mathcal{O}V \cap \mathcal{O}gV = \emptyset$. The set $S = \widetilde{V} \subseteq G$ is open, big on the left, and $\mathcal{O}\widetilde{V} = \widetilde{\mathcal{O}V}$, which implies that $\mathcal{O}S \cap gS = \emptyset$, that is, $g \notin \overline{SS^{-1}}$, as desired.

(iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii): trivial.

(ii) $\Rightarrow$ (iv): Find a symmetric $V \in \mathcal{N}(G)$ with $\overline{SS^{-1}} \cap VgV = \emptyset$ and then a $W \in \mathcal{N}(G)$ such that $W^2 \subseteq V$. One has $WgW \cap (WS)(WS)^{-1} = \emptyset$. Choose a compact K with $KS = G$ and then a finite F with $FW \supseteq K$. Then $F(WS) = (FW)S \supseteq KS = G$, and the set WS is open, big on the left, and $g \notin \overline{(WS)(WS)^{-1}}$.

(iv) $\Rightarrow$ (i): Let $U \in \mathcal{N}(G)$ be symmetric and such that $UgU \cap SS^{-1} = \emptyset$. It means that $gUS \cap US = \emptyset$. Find a right-invariant continuous pseudometric ρ on G with $\rho \leq 1$ and $\mathcal{O}^{\rho}_1(e) \subseteq U$. For any $x \in G$, set $f(x) = \rho\text{-dist}\,(x, G \setminus US)$. Clearly, $0 \leq f(x) \leq 1$. Whenever $xy^{-1} \in \mathcal{O}^{\rho}_{\varepsilon}(e)$, one has by the triangle inequality $|f(x) - f(y)| \leq \rho(x,y) < \varepsilon$, that is, f is in $C^*_{r}(G)$. If $x \in S$ then $f(x) = 1$, and if $x \notin US$ then $f(x) = 0$. Therefore, $f|_{gS} \equiv 0$ and $f|_S \equiv 1$.

Choose a finite $F \subseteq G$ with $FS = G$. Denote by $[\cdot]$ the closure operator in $\mathcal{S}(G)$; since $G = \cup_{\kappa \in F} \kappa S$ and F is finite, one has $\mathcal{S}(G) = \cup_{\kappa \in F}[\kappa S]$. For each $\kappa \in F$, one clearly has $\overline{L_\kappa(f)}|_{[\kappa S]} \equiv 1$ and $\overline{L_\kappa(f)}|_{(\kappa g \kappa^{-1})[\kappa S]} \equiv 0$, where the horizontal bar denotes the unique continuous extension of a $\mathcal{U}_{r}(G)$-u.c. bounded function to $\mathcal{S}(G)$.

For at least one $\kappa \in F$ one has $[\kappa S] \cap \mathcal{M}(G) \neq \emptyset$, where $\mathcal{M}(G)$ is identified with a subflow of the greatest ambit. Choose an x from the latter intersection. The property from the previous paragraph implies that $(\kappa g \kappa^{-1})x \neq x$ (because these two points are separated by the function $\overline{L_\kappa(f)}$) and therefore $g(\kappa^{-1}x) \neq \kappa^{-1}x$, where $\kappa^{-1}x \in \mathcal{M}(G)$. $\square$

8. Trivial Universal Minimal Flows

Let us now examine the issue of (non)triviality of the universal minimal flow. A topological semigroup S is called *extremely (left) amenable* [Gr1, Mi, Gr2] if every compact S-flow has a fixed point. For topological groups this amounts to saying that $\mathcal{M}(G) = \{*\}$ is a singleton. This property is a considerably strengthened version of amenability, as can be seen from a well-known characterization of amenable topological groups as those whose every continuous action upon a convex compact set by affine transformations has a fixed point [G1]. (There are many descriptions of extremely amenable topological semigroups, including even one in terms of geometry of Banach spaces [Gr2].) While examples of extremely amenable topological semigroups abound, the problem of existence of extremely amenable topological

groups, explicitely stated by Mitchell in 1970 [Mi], happened to be much more difficult. The first such example was published in 1974 by Herer and Christensen [HR], and in 1996 more examples with different combinations of properties were published by Eli Glasner [G3], who kept his construction unpublished for a while and observed that it was obtained independently by Furstenberg and B. Weiss, and by the present author [P].

Granirer proposed in 1967 [Gr1] the problem of characterizing extremely amenable topological groups in intrinsic terms. We suggest the following solution, which is an immediate corollary (and in fact is an equivalent form of) Theorem 7.1.

THEOREM 8.1 (Extreme Amenability Criterion). *A topological group G is extremely amenable if and only if whenever $S \subseteq G$ is big on the left, SS^{-1} is everywhere dense in G.* □

COROLLARY 8.2 (Glasner [G3]). *If there exists a monothetic minimally almost periodic topological group G that is not extremely amenable, then there exists a big subset $S \subseteq \mathbb{Z}$ such that $S - S$ is not a Bohr neighbourhood of zero.*

PROOF. Minimal almost periodicity of a topological group is obviously equivalent to the property: every neighbourhood of identity in the finest precompact topology on the underlying discrete group is everywhere dense in the original topology on G. Now it is enough to apply Theorem 8.1 to an everywhere dense copy of $\mathbb{Z}$. □

The question of existence of a monothetic minimally almost periodic non-extremely amenable topological group, suggested by Glasner, remains open, as is the classical question on the existence of a big set, S, of integers such that $S - S$ is not a Bohr neighbourhood of zero. (Cf. [G3, V1].)

9. Extreme Amenability of Levy Groups

The following is a slight extension of a definition from [G3], where we drop the metrizability restriction. We say that a topological group G is a *Levy group* if G contains an increasing chain of compact subgroups $G_i, i \in \mathbb{N}$ with the everywhere dense union and such that whenever $A_i \subseteq G_i$ have the property that $\liminf \mu_i(A_i) > 0$, then for every neighbourhood of identity, V, $\lim \mu_i(VA_i \cap G_i) = 1$, where μ_i denotes the normalized Haar measure on G_i.

An example of a Levy group (due to Glasner and, independently, Fusternberg and Weiss) is the group of all measurable $\mathbb{S}^1$-valued complex functions on a nonatomic Lebesgue measure space X equipped with the metric

$$d(f,g) = \int |f(x) - g(x)| d\mu(x)$$

Here one can choose as the compact subgroups G_i tori of increasing dimension formed by step functions corresponding to a sequence of refining partitions of X. A subtle argument [G3] shows that this group is also monothetic.

The following was proved by Gromov and Milman [GM] under the additional condition of equicontinuity of the action, later removed by Glasner [G3] and (independently, unpublished) by Furstenberg and B. Weiss. (The assumed metrizability of the group is also redundant.)

THEOREM 9.1. *Every Levy group is extremely amenable.*

PROOF. Let S be any subset big on the left, and let F be finite with $FS = G$; one can assume that $F \subseteq \cup G_i$. Then $\liminf \mu_i(G_i \cap S) \geq 1/n$, where $n = |F|$. (Indeed, $\mu_i(G_i \cap S) \geq 1/n$ whenever $F \subseteq G_i$.) Since G is a Levy group, for any neighbourhood of identity, V, one has $\lim \mu_i(V(G_i \cap S) \cap G_i) = 1$. For all i large enough, $\mu_i(((V(G_i \cap S) \cap G_i) \cdot (S^{-1} \cap G_i))) = 1$, since assuming the contrary implies that for a confinal set of i, a suitable translate of $G_i \cap S$ in G_i, having measure $\geq 1/n$, does not meet the set $V(G_i \cap S) \cap G_i$ of measure $> 1 - 1/n$, a contradiction. Since $VSS^{-1} \cap G_i \supseteq (V(G_i \cap S) \cap G_i) \cdot (S^{-1} \cap G_i))$, the set $VSS^{-1} \cap G_i$ has full measure and is everywhere dense in G_i starting from some i and thus VSS^{-1} is everywhere dense in G. Since V is arbitrary, SS^{-1} is everywhere dense in G, and Theorem 8.1 finishes the proof. □

10. Extreme Amenability of Groups of Order Automorphisms

Let us say that a group G of order automorphisms of a linearly ordered set X is *ω-transitive* if it takes any finite subset to any other subset of the same size.

THEOREM 10.1 (Pestov [P]). *An ω-transitive group of order automorphisms of an infinite linearly ordered set X, equipped with the topology of simple convergence on X, is extremely amenable.*

PROOF. Stabilizers, St_M, of finite subsets $M \subseteq X$ are open subgroups forming a neighbourhood basis at identity, and the right factor space G/St_M can be naturally identified with the collection $X^{(n)}$ of all ordered n-subsets of X, where $n = |M|$. Let $F \subseteq G$ be finite and such that $FS = G$. According to Infinite Ramsey's Theorem [GRS], for some infinite $A \subseteq X$ and some $f \in F$ one has $A^{(n)} \subseteq \pi_M(fS)$, where $\pi_M\colon G \to X^{(n)}$ is the factor-map. The set $B = f^{-1}(A)$ is also infinite and $B^{(n)} \subseteq \pi_M(S)$. In other words, $\mathrm{St}_M S$ contains all transformations taking M to a subset of B. Consequently, $S^{-1}\mathrm{St}_M$ contains all transformations such that the image of B contains M. Since $B \setminus M$ is infinite, any transformation from G can be represented as a composition of one from $S^{-1}\mathrm{St}_M$ and one from $\mathrm{St}_M S$, that is, $\mathrm{St}_M SS^{-1}\mathrm{St}_M = G$. Since M is arbitrary, it means that $G = \overline{SS^{-1}}$, as required. □

An example of such a topological group is the group $\mathrm{Aut}\,(\mathbb{Q})$ of all order automorphisms of the rational numbers, equipped with the topology of pointwise convergence on the discrete set $\mathbb{Q}$. Since this group naturally embeds as a topological subgroup into the unitary group $\mathrm{U}(l_2(|\mathbb{Q}|))$ with the strong operator topology, we conclude that the unitary group of an infinite-dimensional Hilbert space admits no free compact flow. The same conclusion and for the same reason holds for the full group of permutations, $\mathrm{Sym}\,(X)$, of an infinite set X equipped with the topology of simple convergence.

A further series of examples is obtained by observing that images of extremely amenable topological groups under continuous homomorphisms onto are extremely amenable. This is how one deduces extreme amenability of the groups of orientiation-preserving homeomorphisms endowed with the compact-open topology, $\mathrm{Homeo}_+(I)$, $\mathrm{Homeo}_+\mathbb{R}$, and the stabilizer of any point $s \in \mathbb{S}^1$ in $\mathrm{Homeo}_+(\mathbb{S}^1)$.

From the latter fact it is easy to deduce that $\mathbb{S}^1$ forms the universal minimal flow for the topological group $\mathrm{Homeo}_+(\mathbb{S}^1)$, and thus we arrive at an example showing

that a topological group can act on the universal minimal flow effectively but not freely.

Since the free group on two generators, F_2, embeds into the multiplicative group of a countable linearly ordered skewfield [N], and any such skewfield is order isomorphic to $\mathbb{Q}$, one can easily deduce that F_2 embeds into $\mathrm{Aut}\,(\mathbb{Q})$ as a closed discrete topological subgroup. It means that a closed topological subgroup of an amenable (even extremely amenable) topological group need not be amenable, unlike it is in the locally compact case (cf. [Gre]).

CONJECTURE 10.2. *Every topological group is isomorphic to a topological subgroup of an extremely amenable topological group.* $\boxed{?}$

All the above examples are presented in detail in [P].

11. Enveloping Semigroup of a Flow and Ellis' Problem

One of the guiding problems of topological dynamics is that of decomposing generic G-flows into some basic, simpler building blocks. Many important structure theorems known in topological dynamics serve this purpose. Here we show that a 1969 problem by Ellis on the enveloping semigroup of the universal minimal flow is of the same kind, and discuss its solution.

The following construction belongs to Ellis [El, E3]. Let G be a topological group, and let X be a compact G-flow. The *enveloping semigroup*, $\mathcal{E}(X)$, of X is a G-ambit defined as follows. Recall that every element $g \in G$ determines a self-homeomorphism, τ_g, of X, called the *g-motion.* The underlying topological space of $\mathcal{E}(X)$ is the closure of the collection of all g-motions, $g \in G$, in X^X equipped with the topology of simple convergence. Since X^X is compact, so is $\mathcal{E}(X)$. Since every element of $\mathcal{E}(X)$ is a (not necessarily continuous) map $X \to X$, a natural left action of G upon $\mathcal{E}(X)$ is determined by composing an $f \in \mathcal{E}(X)$ with the g-motion on the left:

$$\mathcal{E}(X) \ni f \overset{g}{\mapsto} \tau_g \circ f \in \mathcal{E}(X)$$

The correctness of the definition and the continuity of the action $G \times \mathcal{E}(X) \to \mathcal{E}(X)$ are checked easily. The identity map Id_X has an everywhere dense orbit in $\mathcal{E}(X)$ (it is simply the set of all motions), and therefore $\mathcal{E}(X) = (\mathcal{E}(X), \mathrm{Id}_X)$ is a G-ambit. Moreover, $\mathcal{E}(X)$ supports an obvious natural semigroup structure, but we are not interested in it.

The following explains the significance of the enveloping semigroup.

THEOREM 11.1. *The enveloping semigroup $\mathcal{E}(X)$ of a G-flow X is the greatest G-ambit with the property that morphisms into X separate points. In other words, morphisms of G-flows $\mathcal{E}(X) \to X$ separate points in $\mathcal{E}(X)$, and whenever (Z, z) is a G-ambit such that morphisms of G-flows $Z \to X$ separate points in Z, there exists a unique morphism of G-ambits $(\mathcal{E}(X), \mathrm{Id}_X) \to (Z, z)$.*

PROOF. The first statement is easy to verify: if $f, g \in \mathcal{E}(X)$ and $f \neq g$, then for some $x \in X$ one must have $f(x) \neq g(x)$ and therefore $\hat{x}(f) \neq \hat{x}(g)$, where $\hat{x}\colon \mathcal{E}(X) \to X$ is the evaluation map at x. It remains to observe that $\hat{x}$ is continuous (by the very definition of the topology of simple convergence) and equivariant (by the definition of composition of mappings), that is, a morphism of G-flows.

Now assume (Z, z) is a G-ambit whose points are separated by continuous equivariant maps to X. Denote by $\mathcal{T}$ the collection of all such maps $Z \to X$. Any

$f \in \mathcal{F}$ is uniquely determined by the image of z, and thus the map

$$\mathcal{F} \ni f \mapsto f(z) \in X$$

is an injection, identifying $\mathcal{F}$ with an everywhere dense subset of X. To every $a \in Z$, associate an element of $X^{\mathcal{F}}$ of the form $f \mapsto f(a)$; the resulting mapping $Z \to X^{\mathcal{F}}$ is one-to-one and continuous and therefore (as X is compact) a homeomorphic embedding. Finally, it is easy to verify that the natural projection $X^X \to X^{\mathcal{F}}$, dual to the inclusion $\mathcal{F} \hookrightarrow X$, determines a continuous equivariant map from $\mathcal{E}(X) \subseteq X^X$ onto Z. □

EXAMPLES 11.2. 1. The enveloping semigroup $\mathcal{E}(\mathcal{S}(G))$ of the greatest ambit of a topological group G is canonically isomorphic to the greatest ambit $\mathcal{S}(G)$ itself.

2. Here is a more interesting case where Theorem 11.1 applies. Let G be a discrete group. The *shift system* over G is topologically a (compact, zero-dimensional) Cantor cube $\mathbb{Z}_2^G$, upon which G acts by left translations. Such G-flows form the subject of study of *symbolic dynamics.* It is shown [G2, Lemma 4.1] that, rather remarkably, the enveloping semigroup $\mathcal{E}(\mathbb{Z}_2^G)$ is isomorphic to $\mathcal{S}(G)$ and therefore the shift system in a sense carries the complete dynamical information pertaining to a discrete group. ▲

COROLLARY 11.3. *For a topological group G, the following are equivalent.*

(i) *The canonical morphism $\mathcal{S}(G) \to \mathcal{E}(\mathcal{M}(G))$ is an isomorphism of G-ambits.*

(ii) *Points of $\mathcal{S}(G)$ are separated by continuous equivariant mappings to minimal G-flows.*

PROOF. $\Rightarrow$: follows from Theorem 11.1. $\Leftarrow$: Let $x, y \in \mathcal{S}(G)$ be arbitrary, and let them be separated by a continuous equivariant mapping, f, to a minimal G-flow, X. Fix a morphism $i\colon \mathcal{M}(G) \to X$, which is necessarily onto, and choose any $a \in i^{-1}(f(e))$. There exists a unique morphism of G-flows $\varphi\colon \mathcal{S}(G) \to \mathcal{M}(G)$ sending e to a. Since $i(\varphi(e)) = f(e)$ and therefore $i \circ \varphi = f$, one has $\varphi(x) \neq \varphi(y)$, as required. □

Ellis asked in 1969 [E3] whether or not the condition (i) was true for every topological group G. In view of the above result, it amounts to asking whether or not the greatest ambit of a topological group can be reconstructed using only minimal G-flows, that is, is 'residually minimal.' Now that the existence of extremely amenable groups has been established, it is clear that in general Ellis' question is answered in the negative. Indeed, while the greatest ambit $\mathcal{S}(G)$ is always nontrivial whenever so is G, the universal minimal flow of an extremely amenable topological group is a singleton and maps to it do not separate points.

The answer to Ellis' question is positive for every precompact group G, because, as we observed earlier, in this case $\mathcal{M}(G)$ coincides with all of $\mathcal{S}(G)$ and is isomorphic, as a G-flow, to the compact group completion of G equipped with the natural left action of G by translations. Surprisingly, it remains the only known class of topological groups answering Ellis' question in the positive. The limitations of space prevent us from presenting a proof of the following recent result by Glasner [G4], since this observation is based on deep results [F, GW].

THEOREM 11.4. *For the group $\mathbb{Z}$ with the discrete topology, the points of the greatest ambit $\mathcal{S}(\mathbb{Z}) \cong \beta\mathbb{Z}$ are not separated by continuous equivariant maps to the universal minimal flow $\mathcal{M}(\mathbb{Z})$.* □

In view of this result, we put forward the following

CONJECTURE 11.5. For a topological group G the following are equivalent: (i) continuous equivariant maps to minimal G-flows separate points in the greatest G-ambit, (ii) G is precompact. $\boxed{?}$

12. Effective Minimal Flows

The following result, deduced at once from Theorem 7.1, is in fact an equivalent twin of Theorem 8.1, recovered from it by applying the latter result to the image of a topological group G in $\mathrm{Homeo}\,(\mathcal{M}(G))$ under the action homomorphism.

THEOREM 12.1. *A topological group G acts effectively on its universal minimal flow $\mathcal{M}(G)$ if and only if for every $g \in G$, $g \neq e$, there is a left syndetic subset $S \subseteq G$ such that $g \notin \overline{SS^{-1}}$.* $\square$

As the first application of this criterion, we are able to offer a simple proof of the following weakened form of Veech's theorem.

THEOREM 12.2. *A locally compact group acts effectively on its universal minimal flow.*

PROOF. Let G be a locally compact group, let $g \in G$ and $g \neq e$. We will verify the condition from Theorem 12.1. Fix a compact neighbourhood of g, say V. Assume $V \not\ni e$. Zorn's Lemma implies the existence of a maximal subset $A \subseteq G$ with $e \in A$ and $AA^{-1} \cap V = \emptyset$. We claim that $(V^{-1} \cup V)A = G$ and in particular A is left syndetic, which finishes the proof.

Assuming it is not so, there is some $g \in G$ with

$$g \notin (V^{-1} \cup V)A \equiv V^{-1}A \cup VA,$$

which means that $Ag^{-1} \cap V = \emptyset = gA^{-1} \cap V$. Now we have

$$(A \cup \{g\})(A \cup \{g\})^{-1} = (AA^{-1} \cap V) \cup (Ag^{-1} \cap V) \cup (gA^{-1} \cap V) \cup (\{e\} \cap A) = \emptyset,$$

in contradiction with the assumed maximality of A. $\square$

PROBLEM 12.3. Describe in intrinsic terms those topological groups acting *freely* on their universal minimal flows and use this description to obtain a 'soft' proof of Veech's theorem. $\boxed{?}$

Remember that $\mathrm{Homeo}_+(\mathbb{S}^1)$ provides an example of a topological group whose action on the universal minimal flow is effective but not free (Section 11).

13. On the Existence of Unitary Representations

Out of multitude of ways to define an amenable topological group [G1, Gre], the one most apt for abstract topological dynamics is this: a topological group G is amenable if and only if every compact G-flow X admits an invariant mean, that is, a positive linear functional $\varphi\colon C(X) \to \mathbb{C}$ of norm 1 such that $\varphi(L_g f) = \varphi(f)$ for all $f \in C(X)$, $g \in G$. (Cf. e.g. [A].)

The following is a somewhat strengthened form of Theorem 4 from [HR].

THEOREM 13.1. *Let an amenable topological group G act effectively on the universal minimal flow $\mathcal{M}(G)$. Then G admits a faithful strongly continuous representation in a Hilbert space.*

PROOF. Let $\varphi\colon C(\mathcal{M}(G)) \to \mathbb{C}$ be an invariant mean. Set for all $f, h \in C(\mathcal{M}(G))$: $\langle f, g\rangle = \varphi(f\bar{h})$. Clearly, $\langle , \rangle$ is an invariant sesquilinear form on $C(\mathcal{M}(G))$. Denote by $\mathcal{H} = \widehat{C(\mathcal{M}(G))}/\mathcal{N}_\varphi$ the associated Hilbert space, where $\mathcal{N}_\varphi = \{x \in C(\mathcal{M}(G))\colon \langle x, x\rangle = 0\}$. The group G acts on $\mathcal{H}$ by isometries. This action is strongly continuous. Indeed, so is the action of G on $C(\mathcal{M}(G))$ and since the pre-Hilbert topology on $C(\mathcal{M}(G))$ is coarser than the uniform topology, each orbit map $G \ni g \mapsto g \cdot f \in \mathcal{H}$ is continuous whenever $f \in C(\mathcal{M}(G))$. It means that the representation of G in $\mathcal{H}$ is continuous with respect to the topology of simple convergence on an everywhere dense subset of $\mathcal{H}$. But on the unitary group $U(\mathcal{H})$ any such topology coincides with the strong operator topology.

It remains to show that the representation is faithful. Let $g \in G$ be any, and let $x \in \mathcal{M}(G)$ be such that $gx \neq x$. Fix a neighbourhood $V \ni x$ with $gV \cap V = \emptyset$. There exists an $f \in C(\mathcal{M}(G))$ with $0 \leq f \leq 1$, $\operatorname{supp} f \subset V$, and $\varphi(f) > 0$. Indeed, otherwise the support of the invariant mean φ would not meet any translate gV, and since finitely many of those cover $\mathcal{M}(G)$ by force of its minimality, the support of φ would be empty, in contradiction to $\| \varphi \| = 1$. Now it is easy to see that $\| f - g \cdot f \| = 2\| f \| > 0$, which means that g as an operator in $\mathcal{H}$ is different from identity. $\square$

Inverting the above result, one concludes that an amenable topological group admitting no nontrivial strongly continuous unitary representations is extremely amenable. This observation (made in the particular case of abelian topological groups, which are all known to be amenable) was employed in [HR] by Herer and Christensen, who constructed an abelian topological group without nontrivial unitary representations and deduced its extreme amenability.

Combining Theorems 13.1 and 7.1 leads to the following new existence result for unitary representations.

COROLLARY 13.2. *Let G be an amenable topological group and let $g \in G$ be such that for some left syndetic subset $S \subseteq G$ one has $g \notin \overline{SS^{-1}}$. Then g is separated from identity by a strongly continuous unitary representation of G.*

PROOF. Denote the action of G on $\mathcal{M}(G)$ by τ. According to Theorem 7.1, $\tau(g) \neq e$ in the topological group $\tau(G) < \operatorname{Homeo}(\mathcal{M}(G))$. Since the latter group acts effectively and minimally on $\mathcal{M}(G)$, it admits a faithful unitary representation π by 13.1, and the unitary representation $\pi \circ \tau$ of G is strongly continuous and separates g from identity. $\square$

Unfortunately, the above result falls short of a criterion, since even for amenable topological groups the existence of such a left syndetic set S is not necessary for the existence of a nontrivial unitary representation. Indeed, as observed in [G4], the monothetic extremely amenable group from Glasner–Furstenberg–Weiss example (Section 10) embeds as a topological subgroup into the unitary group of l_2 with the strong operator topology. Thus, the old problem of describing in intrinsic terms those topological groups admitting a complete system of strongly continuous unitary representations remains open.

In view of Herer–Christensen's example [HC], the following curious question seems to be unanswered.

QUESTION 13.3. Does there exist a *monothetic* topological group admitting no nontrivial strongly continuous unitary representations in a Hilbert space? $\boxed{?}$

The importance of big sets in abstract harmonic analysis and representation theory is underlined by the following classical result.

THEOREM 13.4 (Cotlar and Ricabarra [CR]). *Let G be an abelian topological group. An element $g \in G$ is separated from identity by a continuous character if and only if there exists a big symmetric open set $S \subseteq G$ with $g \notin S^6$.* $\square$

Later Ellis and Keynes [EK] reduced the number 6 to 4.

In this connection, the following might be interesting.

CONJECTURE 13.5. *Let G be a topological group. An element $g \in G$ is separated from identity by a continuous finite-dimensional unitary representation if and only if there exists a big symmetric open set $S \subseteq G$ with $g \notin S^4$ (or S^6, etc.).* $\boxed{?}$

Of course, the necessity ($\Rightarrow$) is always valid. Possibly, one extra condition to be imposed on S is that of being *invariant* under inner automorphisms of G.

References

[A] J. Auslander, *Minimal Flows and Their Extensions*, North-Holland Mathematics Studies **153**, North-Holland, Amsterdam–NY–London–Tokyo, 1988.

[B] B. Beauzamy, *Introduction to Operator Theory and Invariant Subspaces*, North-Holland Mathematics Library **42**, North-Holland, Amsterdam–NY–Oxford–Tokyo, 1988.

[BP] C. Bessaga and A. Pełczyński, *Selected Topics in Infinite-Dimensional Topology*, PWN, Warszawa, 1975.

[Br] R.B. Brook, *A constructions of the greatest ambit*, Math. Systems Theory **4** (1970), 243–248.

[CR] M. Cotlar and R. Ricabarra, *On the existence of characters in topological groups*, Amer. J. Math. **76** (1954), 375–388.

[DPS] D. Dikranjan, I. Prodanov, and L. Stoyanov, *Topological Groups. Characters, Dualities, and Minimal Group Topologies*, Monographs and Textbooks in Pure and Applied Mathematics **130**, Marcel Dekker, Inc., NY–Basel, 1989.

[E1] R. Ellis, *A semigroup associated with a transformation group*, Trans. Amer. Math. Soc. **94** (1960), 272–281.

[E2] ———, *Universal minimal sets*, Proc. Amer. Math. Soc. **11** (1960), 540–543.

[E3] ———, *Lectures on Topological Dynamics*, Mathematical Lecture Note Series, W.A. Benjamin Inc., New York, 1969.

[EK] R. Ellis and H.B. Keynes, *Bohr compactifications and a result of Følner*, Israel J. Math. **12** (1972), 314–330.

[En] R. Engelking, *General Topology*, PWN, Warczawa, 1977.

[F] H. Furstenberg, *Disjointness in Ergodic Theory, minimal sets and a problem of diophantine approximation*, Math. System Theory **1** (1967), 1-55.

[G1] S. Glasner, *Proximal Flows*, Lecture Notes in Math. **517**, Springer-Verlag, Berlin–Heidelberg–NY, 1976.

[G2] ———, *Divisibility properties and the Stone-Cech compactification*, Canadian J. Math **32** (1980), 993-1007.

[G3] ———, *Structure theory as a tool in topological dynamics*, Lectures given during the Descriptive Set Theory and Ergodic Theory Joint Workshop (Luminy, June 1996), Tel Aviv University preprint, 26 pp.

[G4] ———, *On minimal actions of Polish groups*, Tel Aviv University preprint, October 1996; to appear in Proc. 1996 Prague Topol. Symposium, 6 pp.

[GW] S. Glasner and B. Weiss, *Interpolation sets for subalgebras of* $1^\infty(\mathbb{Z})$, Israel J. Math. **44** (1983), 345-360.

[GRS] R.L. Graham, B.L. Rothschild, and J.H. Spencer, *Ramsey Theory*, Wiley-Interscience, NY, 1980.

[Gr1] E. Granirer, *Extremely amenable semigroups 2*, Math. Scand. **20** (1967), 93-113.
[Gr2] ———, *Functional analytic properties of extremely amenable semigroups*, Trans. Amer. Math. Soc. **137** (1969), 53–75.
[Gre] F.P. Greenleaf, *Invariant Means on Topological Groups*, Van Nostrand Mathematical Studies **16**, Van Nostrand – Reinhold Co., NY–Toronto–London–Melbourne, 1969.
[GM] M. Gromov and V.D. Milman, *A topological application of the isoperimetric inequality*, Amer. J. Math. **105** (1983), 843–854.
[Gu] A. Guichardet, *Systèmes dynamiques non commutatifs*, Asterisque **13-14**, 1974.
[HC] W. Herer and J.P.R. Christensen, *On the existence of pathological submeasures and the construction of exotic topological groups*, Math. Ann. **213** (1975), 203–210.
[Ma] R.D. Mauldin (ed.), *The Scottish Book*, Birkhäuser, Boston–Basel–Stuttgart, 1981.
[Mi] T. Mitchell, *Topological semigroups and fixed points*, Illinois J. Math.. **14** (1970), 630-641.
[N] B.H. Neumann, *On ordered division rings*, Trans. Amer. Math. Soc. **66** (1949), 202–252.
[P] V.G. Pestov, *On free actions, minimal flows, and a problem by Ellis*, Research report RP-96-193, Dept. Math., Victoria Univ. of Wellington, August 1996, 15 pp. (Available at `http://www.vuw.ac.nz/~vova/papers/ellis.tex`).
[T] S. Teleman, *Sur la représentation linéaire des groupes topologiques*, Ann. Sci. Ecole Norm. Sup. **74** (1957), 319–339.
[Tu] S. Turek, *Universal minimal dynamical system for reals*, Comment. Math. Univ. Carol. **36** (1995), 371–375.
[U] V.V. Uspenskiĭ, *A universal topological group with countable base*, Funct. Anal. Appl. **20** (1986), 160–161.
[V1] W.A. Veech, *The equicontinuous structure relation for minimal abelian transformation groups*, Amer. J. Math. **90** (1968), 723–732.
[V2] ———, *Topological dynamics*, Bull. Amer. Math. Soc. **83** (1977), 775–830.
[Vr] J. de Vries, *Elements of topological dynamics*, Mathematics and Its Applications **257**, Kluwer Academic Publ., Dordrecht–Boston–London, 1993.

School of Mathematical and Computing Sciences, Victoria University of Wellington, P.O. Box 600, Wellington, New Zealand
E-mail address: `vladimir.pestov@mcs.vuw.ac.nz`